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Quantum algebras associated with Bell states

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Abstract
The Bell matrix has become an interesting interdisciplinary topic involving quantum information theory and the Yang–Baxter equation. It is an antisymmetric unitary solution of the braided Yang–Baxter equation and yields all the Bell states by acting on the product basis. In this paper, using the Faddeev–Reshetikhin–Takhtadjian (FRT) construction, we obtain a quantum algebra associated with the Bell matrix. We explore two characteristic algebraic structures in its four-dimensional representation. One is a representation with a composition series, namely, it has irreducible subrepresentations but is not completely reducible. The other is a direct sum of two-dimensional cyclic representations, and can be spanned by four maximally entangled states as local unitary transformations of the Bell states. Both of them are expected to be realized in physical systems and exploited in quantum information theory. Besides, we present the other quantum algebra associated with the unitary evolution of the Bell states (or the Yang–Baxterization of the Bell matrix).

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1. Introduction

The present paper is one in a series of recent papers [1–8] that attempt to set up natural connections between quantum information theory [9] and the Yang–Baxter equation (YBE) [10, 11]. These connections are expected to be helpful for solving problems in quantum information theory as well as developing the YBE theory. Quantum information theory is a fast growing new scientific subject and combines fundamentals of quantum mechanics with modern computer science. It mainly deals with the problem of how to store and operate physical information to perform an assigned task (such as quantum computation in a quantum computer). The YBE was originally found in the procedure of looking for exact solutions of two-dimensional quantum field theories or lattice models in statistical physics. It has been a well-developed discipline in the sense of both physics (integrable models [12]) and mathematics (quantum groups [13]).

There are natural similarities between quantum entanglements [14] and topological entanglements [15]. Quantum entanglements describe that some quantum states in quantum mechanics cannot be a tensor product of another two quantum states. Topological entanglements denote topological configurations like links or knots which are closures of braids. Aravind [16] observed that it is possible to identify a quantum measurement of the quantum state with deleting one component of the link. But this type of correspondence between quantum states and topological links is not basis-independent. A deeper method (we believe) suggested by Kauffman and Lomonaco [2] is to consider unitary quantum gates $\check{R}$ that are both universal for quantum computation and are also the solutions to the condition for topological braiding. Such $\check{R}$-matrices are unitary solutions to the braided YBE, and can be used as universal quantum gates which have an entangling power of transforming a separate quantum state to an entangled one. Moreover, in topological quantum computing proposed by Kitaev [17] and Freedman *et al* [18], unitary braids are exploited as logic gates to operate quasiparticles called anyons, namely, quantum information is processed by braiding anyons. In condensed matter physics, Abelian or non-Abelian anyons can be created in the fractional quantum Hall effect [19]. Topological quantum computation is a very appealing approach for experimentally realizing a physical quantum computer.

The Bell matrix has become an interesting interdisciplinary topic involving quantum information theory and the Yang–Baxter equation. The observation that all Bell states can be generated by the Bell matrix acting on the product basis has been stimulating the recent study: the Bell matrix is identified with a universal quantum gate [1, 2]; Yang–Baxterization of the Bell matrix [3, 4] is used to derive the Hamiltonian determining the evolution of the Bell states; the braid teleportation configuration [6, 7] in terms of the Bell matrix is found to be a sort of algebraic structure underlying the quantum teleportation. The Bell states play crucial roles in both fundamental problems and practical applications of quantum information theory [9]. They are defined by various known entanglement measures to be bipartite maximally entangled, and have been widely exploited in various topics of quantum information such as quantum entanglements [14], quantum cryptography [20] and quantum teleportation [21]. The Bell theorem [22] describes incompatibility between quantum theory and classical deterministic local models in the form of the Bell inequalities among various statistical correlations.

As a solution of the braided YBE [10, 11], the Bell matrix forms a unitary braid representation and can be used to calculate the Jones polynomial [5]. But it does not get involved in the previous study of integral models [12] in some two-dimensional quantum field theories and statistical physics). Matrix entries of symmetric solutions of eight-vertex models [11] are either positive or zero and are explained as the Boltzmann weights in statistical physics, whereas the Bell matrix has negative matrix entries which cannot be the Boltzmann weights. Here for convenience we call the Bell matrix antisymmetric solution of eight-vertex models, since it also has eight non-vanishing matrix entries and is non-triangular and non-singular. Note that in the literature the term 'eight-vertex model' was a pronoun presenting the original model solved by Baxter [11, 23, 24].

In view of the previous work that a new quantum group can be found via a 'non-standard' braid group representation [25], in this paper we study the quantum algebra obtained by applying Faddeev–Reshetikhin–Takhtadjian (FRT) construction [26] to the Bell matrix. We explore two characteristic algebraic structures in its four-dimensional representation. The one is a representation with a composition series, namely, it has irreducible subrepresentations but is not completely reducible. The other is a direct sum of the two-dimensional cyclic representations and can be spanned by four maximally entangled states as local unitary transformations of the Bell states. Both of them are expected to be realized in a physical system and exploited in quantum information theory. Besides, we determine the unitary evolution of Bell states via Yang–Baxterization [27, 28] and derive the associated quantum algebra with the FRT construction.

The plan of this paper is organized as follows. Section 2 introduces the braided YBE, Bell states and the Bell matrix as a universal quantum gate. Section 3 presents the quantum algebra associated with the Bell states. Section 4 shows its all two-dimensional representations and focuses on two specific examples characterizing its four-dimensional representation. Section 5 discusses the quantum algebra related to the unitary evolution of Bell states. The last section comments on physical realizations of the Bell matrix and associated quantum algebras as well as their applications to quantum information and computation.

2. The Bell matrix and a universal quantum gate

The Bell matrix forms a unitary braid representation, generates the Bell states from the product basis and can be identified with a universal quantum gate in quantum computation. Hence it becomes an interesting topic involving both quantum information and the YBE.

2.1. The braided Yang–Baxter equation

The YBE without the spectral parameter is called the braided YBE, i.e., the braid group relation describing low-dimensional topology. Artin's braid group $\mathcal{B}_n$ on n strands has the well-known presentation in terms of generators $b_1, \ldots, b_{n-1}$ satisfying the commutation relation

$$b_i b_j = b_j b_i, \qquad |i - j| \geqslant 2 \tag{1}$$

and the braid relations

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}, \qquad 1 \leqslant i \leqslant n-2. \tag{2}$$

Usually, the last relation leads to the braided version of the YBE, i.e.,

$$(\check{R} \otimes \mathbb{1}_d)(\mathbb{1}_d \otimes \check{R})(\check{R} \otimes \mathbb{1}_d) = (\mathbb{1}_d \otimes \check{R})(\check{R} \otimes \mathbb{1}_d)(\mathbb{1}_d \otimes \check{R}), \tag{3}$$

with an invertible $d^2 \otimes d^2$ matrix $\check{R} : V \otimes V \to V \otimes V$ ($V \equiv \mathbb{C}^d$ being a d-dimensional complex vector space). Relation (3) gives rise to a sequence of representations $(\pi_n, (\mathbb{C}^d)^{\otimes n})$ of $\mathcal{B}_n$

$$\pi_n(b_i) = \mathbb{1}_d^{\otimes i-1} \otimes \check{R} \otimes \mathbb{1}_d^{\otimes n-i-1} \tag{4}$$

since clearly $\pi_n(b_i)$ and $\pi_n(b_j)$ commute for $|i - j| \geqslant 2$.

2.2. Bell states and Bell matrix

The two-dimensional unit matrix is denoted by $\mathbb{1}_2$, the Pauli matrices $\sigma_x, \sigma_y, \sigma_z$ have the conventional forms, and the symbols σ_+, σ_- are given by

$$\sigma_+ = \frac{1}{2}(\sigma_x + \mathrm{i}\sigma_y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma_- = \frac{1}{2}(\sigma_x - \mathrm{i}\sigma_y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{5}$$

which are nilpotent operators satisfying $\sigma_+^2 = \sigma_-^2 = 0$.

The two-dimensional Hilbert space $\mathcal{H}_2$ spanned by two eigenvectors $|m\rangle, m = 0, 1$ of the spin-$\frac{1}{2}$ operators (i.e., Pauli matrices, for example, $\sigma_z|0\rangle = |0\rangle, \sigma_z|1\rangle = -|1\rangle$), has a realization of the coordinate vectors over the complex field $\mathbb{C}$,

$$|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad \alpha, \beta \in \mathbb{C}. \tag{6}$$

A state vector in this $\mathcal{H}_2$ is usually called a qubit in quantum information theory [9], and $\mathcal{H}_2 \cong \mathbb{C}^2$. Denote $|ij\rangle, i, j = 0, 1$ for the product basis of a four-dimensional Hilbert space over the complex field $\mathbb{C}$, and the four orthonormal Bell states have the forms

$$|\psi_\pm\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), \qquad |\phi_\pm\rangle = \tfrac{1}{\sqrt{2}}(|10\rangle \pm |01\rangle) \tag{7}$$

which are transformed to each other under local unitary transformations denoted by the Pauli matrices

$$|\psi_-\rangle = (\mathbb{1}_2 \otimes \sigma_z)|\psi_+\rangle, \qquad |\phi_+\rangle = (\mathbb{1}_2 \otimes \sigma_x)|\psi_+\rangle, \qquad |\phi_-\rangle = (\mathbb{1}_2 \otimes -\mathrm{i}\sigma_y)|\psi_+\rangle. \tag{8}$$

There are two Bell matrices, B_+ and B_-, defined in [3, 4, 6, 7]

$$B_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \qquad B_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \tag{9}$$

which have formalisms in exponential functions

$$B_+ = \mathrm{e}^{\mathrm{i}\frac{\pi}{4}(\sigma_y \otimes \sigma_x)}, \qquad B_- = \mathrm{e}^{\mathrm{i}\frac{\pi}{4}(\sigma_x \otimes \sigma_y)}. \tag{10}$$

The Bell matrix can yield all the Bell states by acting on the product basis, for example,

$$B_+|00\rangle = |\psi_-\rangle, \qquad B_+|11\rangle = |\psi_+\rangle, \qquad B_+|01\rangle = |\phi_-\rangle, \qquad B_+|10\rangle = |\phi_+\rangle. \tag{11}$$

See [1, 2], the Bell matrix is found to satisfy the braided YBE (3) with $d = 2$ and forms a unitary braid representation of the braid group $\mathcal{B}_n$.

2.3. The Bell matrix as a universal quantum gate

A two-qubit gate G is a unitary linear mapping $G : V \otimes V \to V$ where V is a complex two-dimensional vector space. We say that the gate G is universal for quantum computation (or just universal) if G together with local unitary transformations (unitary transformations from V to V) generates all unitary transformations of the complex vector space of dimension 2^n to itself. It is well known [9] that the CNOT gate is a universal quantum gate satisfying

$$\begin{aligned} &\mathrm{CNOT}|00\rangle = |00\rangle, \qquad \mathrm{CNOT}|01\rangle = |01\rangle, \\ &\mathrm{CNOT}|10\rangle = |10\rangle, \qquad \mathrm{CNOT}|11\rangle = |10\rangle. \end{aligned} \tag{12}$$

Kauffman and Lomonaco [2] proved that the Bell matrix acts as a universal quantum gate and gives rise to a specific presentation of the CNOT gate by combining with local unitary transformations. For example, consider the Bell matrix B_- and denote four two-dimensional unitary matrices $\alpha, \beta, \gamma, \delta$ by

$$\begin{aligned} &\alpha = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}, \qquad \beta = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ \mathrm{i}/\sqrt{2} & \mathrm{i}/\sqrt{2} \end{pmatrix}, \\ &\gamma = \begin{pmatrix} 1/\sqrt{2} & \mathrm{i}/\sqrt{2} \\ 1/\sqrt{2} & -\mathrm{i}/\sqrt{2} \end{pmatrix}, \qquad \delta = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix}, \end{aligned} \tag{13}$$

and then is straightforward to verify that

$$\mathrm{CNOT} = (\alpha \otimes \beta) \cdot B_- \cdot (-\gamma \otimes \delta).$$

As a remark, with the help of the Bell matrix as the CNOT gate, it is expected to find and generalize quantum polynomially complex algorithm that would eventually speed up the computation on quantum computers.

3. Quantum algebra associated with the Bell matrix

The FRT construction [26] is a standard procedure of obtaining a quantum algebra over the complex field $\mathbb{C}$ associated with the $\check{R}$-matrix, an invertible solution of the braided YBE. All the generators are collected in the T-matrix satisfying the $\check{R}TT$ relation: $\check{R}(T \otimes T) = (T \otimes T)\check{R}$. The FRT construction is originally devised for six-vertex models [12, 13] in which the $\check{R}$-matrix has six non-vanishing matrix entries, whereas the $\check{R}$-matrix in this paper is the Bell matrix with eight non-vanishing matrix entries. Hence our quantum algebra is not completely the same as Hopf algebras and quantum groups [13], also see [29] for a helpful comment on generalized quantum algebras.

Let us start with the $\check{R}_\omega$-matrix, a solution of the YBE without spectral parameter [3, 4],

$$\check{R}_\omega = \begin{pmatrix} 1 & 0 & 0 & q \\ 0 & 1 & 1 & 0 \\ 0 & \omega & 1 & 0 \\ \omega q^{-1} & 0 & 0 & 1 \end{pmatrix}, \qquad \omega = \pm 1, \qquad q \neq 0, q \in \mathbb{C}, \tag{14}$$

where the $\check{R}_{-1}$-matrix is a deformation of the Bell matrix B_+, and the $\check{R}_1$-matrix is a deformation of a symmetric solution of eight-vertex models. For simplicity, the Bell matrix B_- will not be involved because it gives rise to the same quantum algebra as the B_+ matrix.

In this section, we set up a quantum algebra $\mathcal{A}_{-1}$ via the $\check{R}TT$ relation where the R-matrix is $\check{R}_{-1}$. The T-matrix has non-commutative operators $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ as its matrix entries, and its tensor product $T \otimes T$ has the form

$$T = \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix}, \qquad T \otimes T = \begin{pmatrix} \hat{a}\hat{a} & \hat{a}\hat{b} & \hat{b}\hat{a} & \hat{b}\hat{b} \\ \hat{a}\hat{c} & \hat{a}\hat{d} & \hat{b}\hat{c} & \hat{b}\hat{d} \\ \hat{c}\hat{a} & \hat{c}\hat{b} & \hat{d}\hat{a} & \hat{d}\hat{b} \\ \hat{c}\hat{c} & \hat{c}\hat{d} & \hat{d}\hat{c} & \hat{d}\hat{d} \end{pmatrix}, \tag{15}$$

where the tensor symbol $\otimes$ in every matrix entry of $T \otimes T$ has been omitted for convenience. The $\check{R}TT$ relation leads to a quantum algebra over the complex field $\mathbb{C}$ with the generators $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ satisfying algebraic relations,

$$\begin{aligned} &\hat{a}\hat{a} = \hat{d}\hat{d}, && \hat{a}\hat{b} = q\hat{d}\hat{c}, && \hat{b}\hat{b} = \omega q^2\hat{c}\hat{c}, && \hat{a}\hat{c} = q^{-1}\hat{d}\hat{b}, \\ &\hat{a}\hat{d} = \hat{d}\hat{a}, && \hat{b}\hat{a} = \omega q\hat{c}\hat{d}, && \hat{b}\hat{c} = \omega\hat{c}\hat{b}, && \hat{c}\hat{a} = \omega q^{-1}\hat{b}\hat{d}. \end{aligned} \tag{16}$$

In this paper, however, we will study the quantum algebra $\mathcal{A}_\omega$ generated by four generators $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ satisfying the algebraic relations (16) except the following two equations,

$$[F_1] \equiv \hat{c}\hat{a} - \omega q^{-1}\hat{b}\hat{d} = 0, \qquad [F_2] \equiv \hat{a}\hat{c} - q^{-1}\hat{d}\hat{b} = 0, \tag{17}$$

because we can derive the following equations,

$$\begin{aligned} &\hat{a}[F_2] = [F_1]\hat{a} = 0, && \hat{d}[F_2] = [F_1]\hat{d} = 0, \\ &\hat{b}[F_1] = [F_2]\hat{b} = 0, && \hat{c}[F_1] = [F_2]\hat{c} = 0 \end{aligned} \tag{18}$$

in terms of the remaining six algebraic relations. As any one of the four generators $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ is invertible or nilpotent, $[F_1]$ and $[F_2]$ will be obviously vanishing, that is to say: in these cases the $\mathcal{A}_\omega$ algebra is equivalent to the original quantum algebra derived from the $\check{R}TT$ relation.

With the help of a rescaling $q\hat{c} \to \hat{c}$, i.e., the deformation parameter q to be absorbed into a new generator $\hat{c}$, the quantum algebra $\mathcal{A}_\omega$ is generated by $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ satisfying algebraic relations,

$$\begin{array}{lll} \hat{a}\hat{a} = \hat{d}\hat{d}, & \hat{a}\hat{d} = \hat{d}\hat{a}, & \hat{b}\hat{b} = \omega\hat{c}\hat{c}, \\ \hat{b}\hat{c} = \omega\hat{c}\hat{b}, & \hat{a}\hat{b} = \hat{d}\hat{c}, & \hat{b}\hat{a} = \omega\hat{c}\hat{d}. \end{array} \tag{19}$$

The quantum algebra $\mathcal{A}_1$, i.e., $\omega = 1$, presents a known quantum algebra obtained from the $\check{R}TT$ relation of symmetric solutions of eight-vertex models. But the quantum algebra $\mathcal{A}_{-1}$, i.e., $\omega = -1$, is very attractive because the Bell matrix becomes interesting only in the recent study of quantum information.

The quantum algebra $\mathcal{A}_{-1}$ has interesting quotient algebras. As the generator $\hat{a}$ is a complex scalar denoted by $\hat{a} = p\hat{I}$ with the unit $\hat{I}$, the algebra $\mathcal{A}_{-1}$ is reduced to an algebra generated by $\hat{a}, \hat{c}, \hat{d}$ satisfying algebraic relations,

$$\hat{a} = p\hat{I}, \qquad \hat{d}^2 = p^2\hat{I}, \qquad \hat{c}\hat{d} = -\hat{d}\hat{c}, \qquad p \neq 0, \quad p \in \mathbb{C}, \tag{20}$$

where we define a composition operator $\hat{b}$ to denote the operator product $\hat{c}\hat{d}$, i.e., $\hat{b} = p^{-1}\hat{d}\hat{c}$, proved to satisfy $\hat{b}^2 = -\hat{c}^2$ and $\hat{b}\hat{c} = -\hat{c}\hat{b}$. Another interesting quotient algebra is to require $\hat{b}, \hat{c}$ to describe fermions, i.e., they satisfy $\hat{b}^2 = \hat{c}^2 = 0, \hat{b}\hat{c} = -\hat{b}\hat{c}$. Especially, as $\hat{b} = \hat{c}$, they are the same fermion.

The quantum algebra $\mathcal{A}_{-1}$ has one-dimensional representations over the complex field $\mathbb{C}$: $\hat{b} = \hat{c} = 0$ and $\hat{a}, \hat{d}$ are complex numbers satisfying $\hat{a}^2 = \hat{d}^2$, while it also has a one-dimensional representation over the field including non-commutative Grassman numbers: $\hat{b}, \hat{c}$ are the Grassman numbers and $\hat{a}, \hat{d}$ are the complex numbers.

Arnaudon *et al* discussed the exotic bialgebras [30–32] by applying the FRT construction to non-triangular non-singular R-matrices [33]. The one quantum algebra they have found is a quotient algebra of our quantum algebra $\mathcal{A}_{-1}$ associated with Bell states. They aimed to finalize the explicit classification of the matrix bialgebras generated by four elements, however we are exploring a possible application of quantum groups (or algebras) to the present study of quantum information theory.

4. Two- and four-dimensional representations of $\mathcal{A}_{-1}$

We list its all two-dimensional representations of the quantum algebra $\mathcal{A}_{-1}$ over the complex field $\mathbb{C}$ and explore two types of interesting algebraic structures in its four-dimensional representation derived by the coproduct [13] of its generators.

4.1. Two-dimensional representations

The quantum algebra $\mathcal{A}_{-1}$ has two subalgebras formed by $\hat{a}, \hat{d}$ and $\hat{b}, \hat{c}$, respectively. Hence it is convenient to require either $\hat{a}$ $(\hat{d})$ or $\hat{b}$ $(\hat{c})$ to be a diagonal matrix in a two-dimensional representation of $\mathcal{A}_{-1}$. Note that the following two-dimensional representations of $\mathcal{A}_{-1}$ will be found to satisfy $[F_1] = [F_2] = 0$.

As the generator $\hat{a}$ has a non-vanishing eigenvalue λ with two degenerate eigenvectors, the generators $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ have the following two-dimensional representations over the complex field $\mathbb{C}$,

$$\hat{a} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \qquad \hat{d} = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}, \qquad \alpha^2 + \beta\gamma = \lambda^2, \lambda \neq 0, \lambda \in \mathbb{C}$$
$$\hat{c} = \begin{pmatrix} c_1 & c_2 \\ c_3 & -c_1 \end{pmatrix}, \qquad 2c_1\alpha = -c_3\beta - c_2\gamma, \qquad \alpha, \beta, \gamma, c_1, c_2, c_3 \in \mathbb{C}, \tag{21}$$

where the generator $\hat{b}$ is determined by $\hat{b} = \lambda^{-1}\hat{d}\hat{c}$. In this representation, taking $\alpha = 0$, $\beta = \gamma = \lambda$ and $c_1 = \mu, c_2 = c_3 = 0$ leads to a representation in terms of the unit matrix $\mathbb{1}_2$ and Pauli matrices,

$$\hat{a} = \lambda \mathbb{1}_2, \qquad \hat{d} = \lambda\sigma_x, \qquad \hat{b} = -\mathrm{i}\mu\sigma_y, \qquad \hat{c} = \mu\sigma_z, \quad \lambda, \mu \in \mathbb{C}, \tag{22}$$

while taking $\alpha = \lambda = 1, \beta = \gamma = 0$ and $c_1 = 0, c_2 = c_3 = 1$ gives another interesting representation,

$$\hat{a} = \mathbb{1}_2, \qquad \hat{d} = \sigma_z, \qquad \hat{b} = \mathrm{i}\sigma_y, \qquad \hat{c} = \sigma_x. \tag{23}$$

As the generator $\hat{a}$ has two distinct complex eigenvalues, $\lambda_1 \neq \lambda_2$, the two-dimensional representation for the generators $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ over the complex field $\mathbb{C}$ is obtained to be

$$\hat{a} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad \hat{d} = \epsilon \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}, \qquad \hat{b} = \epsilon \begin{pmatrix} 0 & c_2 \\ -c_3 & 0 \end{pmatrix}, \qquad \hat{c} = \begin{pmatrix} 0 & c_2 \\ c_3 & 0 \end{pmatrix}, \tag{24}$$

where the parameter ϵ satisfies $\epsilon^2 = 1$. As $\hat{b}, \hat{c}$ represent the same fermion, i.e., $\epsilon = 1, c_2 = 1$, $c_3 = 0$, we have the two-dimensional representation,

$$\hat{b} = \hat{c} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \hat{a} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad \hat{d} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}. \tag{25}$$

Now we study the two-dimensional representation in which the generator $\hat{b}$ is a diagonal matrix. If $\hat{b}$ has an eigenvalue with two degenerate eigenvectors, i.e., a scalar operator, the generator $\hat{c}$ has to be vanishing. As $\hat{b}$ has two distinct eigenvalues p_1, p_2 with two eigenvectors $\vec{v}_1, \vec{v}_2$, $p_1 = -p_2$ has to be satisfied because $\hat{c}\vec{v}_1$ is an eigenvector of $\hat{b}$ with the eigenvalue $-p_1$. Hence the generators $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ have a non-vanishing two-dimensional representation over $\mathbb{C}$,

$$\hat{b} = p\sigma_z, \qquad \hat{c} = \begin{pmatrix} 0 & -p^2 \\ 1 & 0 \end{pmatrix}, \qquad p \neq 0,$$
$$\hat{a} = \begin{pmatrix} \alpha & p^2\beta \\ \beta & \alpha \end{pmatrix} = \alpha \mathbb{1}_2 + \beta \begin{pmatrix} 0 & p^2 \\ 1 & 0 \end{pmatrix}, \tag{26}$$
$$\hat{d} = \begin{pmatrix} p\beta & p\alpha \\ p^{-1}\alpha & p\beta \end{pmatrix} = p\beta \mathbb{1}_2 + p^{-1}\alpha \begin{pmatrix} 0 & p^2 \\ 1 & 0 \end{pmatrix}, \qquad \beta \neq 0,$$

where $\beta = 0$ gives an example for the case that $\hat{a}$ is a scalar.

4.2. *Four-dimensional representation: composition series*

The coproduct [13] is a linear map Δ from the vector space V to its tensor product $V \otimes V$ satisfying the coassociativity axiom $(\Delta \otimes Id) \circ \Delta = (Id \otimes \Delta) \circ \Delta$, where Id is an identity map. The coproducts of the generators $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ have the forms

$$\Delta(\hat{a}) = \hat{a} \otimes \hat{a}' + \hat{b} \otimes \hat{c}', \qquad \Delta(\hat{b}) = \hat{a} \otimes \hat{b}' + \hat{b} \otimes \hat{d}',$$
$$\Delta(\hat{c}) = \hat{c} \otimes \hat{a}' + \hat{d} \otimes \hat{c}', \qquad \Delta(\hat{d}) = \hat{c} \otimes \hat{b}' + \hat{d} \otimes \hat{d}', \tag{27}$$

where the generators $\hat{a}', \hat{b}', \hat{c}', \hat{d}'$ satisfy the quantum algebra $\mathcal{A}_{-1}$ and play the same roles as $\hat{a}, \hat{b}, \hat{c}, \hat{d}$, respectively. It is easy to prove the above coproducts to satisfy algebraic relations of the quantum algebra $\mathcal{A}_{-1}$, for example,

$$\Delta(\hat{a})\Delta(\hat{a}) = \Delta(\hat{d})\Delta(\hat{d}), \qquad \Delta(\hat{a})\Delta(\hat{d}) = \Delta(\hat{d})\Delta(\hat{a}) \tag{28}$$

and also $\Delta([F_1]) = \Delta([F_1]) = 0$ as $[F_1] = [F_2] = 0$.

Here we derive four-dimensional representations $\underline{4}$ of $\mathcal{A}_{-1}$ via its coproduct structure in terms of its known two-dimensional representations and denote this sort of four-dimensional representation by $\underline{4} = \underline{2} \otimes \underline{2}$. The representation $\underline{4}$ is defined by the coproduct map from the generators $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ to their coproducts $\Delta(\hat{a}), \Delta(\hat{b}), \Delta(\hat{c}), \Delta(\hat{d})$ in the way

$$\hat{a}|ij\rangle \equiv \Delta(\hat{a})|ij\rangle, \qquad \hat{b}|ij\rangle \equiv \Delta(\hat{b})|ij\rangle, \qquad \hat{c}|ij\rangle \equiv \Delta(\hat{c})|ij\rangle, \qquad \hat{d}|ij\rangle \equiv \Delta(\hat{d})|ij\rangle, \tag{29}$$

where $|0\rangle, |1\rangle$ are the bases of $\underline{2}$ and $|ij\rangle, i, j = 0, 1$ are the bases of $\underline{4}$. And these four-dimensional representations of $\mathcal{A}_{-1}$ satisfy $[F_1] = [F_2] = 0$, too.

In the following, we present two examples for its four-dimensional representations. In the first one, we exploit the two-dimensional representation (25) with λ_1 relabeled to be 1 and λ_2 to be λ, i.e.,

$$\begin{aligned} &\hat{a}|0\rangle = |0\rangle, \hat{a}|1\rangle = \lambda|1\rangle, \qquad &&\hat{d}|0\rangle = |0\rangle, \hat{d}|1\rangle = -\lambda|1\rangle, \\ &\hat{b}|0\rangle = 0, \hat{b}|1\rangle = |0\rangle, \qquad &&\hat{c}|0\rangle = 0, \hat{c}|1\rangle = |0\rangle, \end{aligned} \tag{30}$$

and the two-dimensional representation for the generators $\hat{a}', \hat{b}', \hat{c}', \hat{d}'$,

$$\begin{aligned} &\hat{a}'|0\rangle = |0\rangle, \quad \hat{a}'|1\rangle = \lambda'|1\rangle, \qquad &&\hat{d}'|0\rangle = |0\rangle, \quad \hat{d}'|1\rangle = -\lambda'|1\rangle, \\ &\hat{b}'|0\rangle = 0, \hat{b}'|1\rangle = |0\rangle, \qquad &&\hat{c}'|0\rangle = 0, \hat{c}'|1\rangle = |0\rangle. \end{aligned} \tag{31}$$

This four-dimensional representation $\underline{4}$ has a composition series over $\mathbb{C}$,

$$\{0\} \subset \{|00\rangle\} \subset \{|00\rangle, |01\rangle\} \subset \{|00\rangle, |01\rangle|10\rangle\} \subset \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}, \tag{32}$$

where the set $\{0\}$ consists of zero element and the set $\{|00\rangle, |01\rangle\}$ can be replaced by $\{|00\rangle, |10\rangle\}$. An ascending chain of subrepresentations is called a composition series if the successive quotients are irreducible representations. We will see that each term in our composition series is an indecomposable representation (i.e. not a direct summand) and all the nontrivial term is reducible. Namely, the four-dimensional representation has irreducible subrepresentations but is not completely reducible.

Let us explain this in detail. The vector space $\{|00\rangle\}$ forms a one-dimensional irreducible subrepresentation of $\mathcal{A}_{-1}$,

$$\hat{a}|00\rangle = |00\rangle, \qquad \hat{b}|00\rangle = 0, \qquad \hat{c}|00\rangle = 0, \qquad \hat{d}|00\rangle = |00\rangle; \tag{33}$$

the vector space $\{|00\rangle, |01\rangle\}$ forms a two-dimensional subrepresentation of $\mathcal{A}_{-1}$,

$$\hat{a}|01\rangle = \lambda'|01\rangle, \qquad \hat{b}|01\rangle = |00\rangle, \qquad \hat{c}|01\rangle = |00\rangle, \qquad \hat{d}|01\rangle = -\lambda'|01\rangle; \tag{34}$$

and the vector space $\{|00\rangle, |10\rangle\}$ forms another two-dimensional subrepresentation of $\mathcal{A}_{-1}$,

$$\hat{a}|10\rangle = \lambda|10\rangle, \qquad \hat{b}|10\rangle = |00\rangle, \qquad \hat{c}|10\rangle = |00\rangle, \qquad \hat{d}|10\rangle = -\lambda|10\rangle; \tag{35}$$

and so the vector space $\{|00\rangle, |01\rangle, |10\rangle\}$ forms a three-dimensional subrepresentation for the algebra $\mathcal{A}_{-1}$.

See figure 1, every horizontal line represents a state in the four-dimensional representation $\underline{4}$ and every line with an oriented arrow denotes a transition between different states which is caused by the action of a generator.

In the four-dimensional representation $\underline{4} = \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, the actions of all generators on $|11\rangle$ are linear combinations of $|ij\rangle, i, j = 0, 1$,

$$\begin{aligned} &\hat{a}|11\rangle = |00\rangle + \lambda\lambda'|11\rangle, \qquad &&\hat{b}|11\rangle = -(\lambda'|01\rangle - \lambda|10\rangle), \\ &\hat{c}|11\rangle = \lambda'|01\rangle - \lambda|10\rangle, \qquad &&\hat{d}|11\rangle = |00\rangle + \lambda\lambda'|11\rangle, \end{aligned} \tag{36}$$

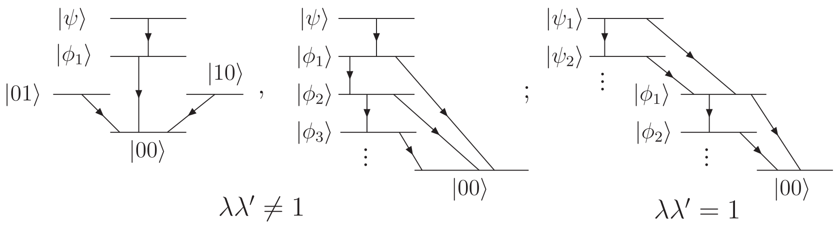

Figure 1. Four-dimensional representation of $\mathcal{A}_{-1}$ with a composition series (32).

and so is impossible for $\underline{4}$ to have completely reducible representations. Similarly, one sees from figure 1 that $\{|00\rangle\}$ is the only nontrivial subrepresentation of $\{|00\rangle, |01\rangle\}$, and hence the representations $\{|00\rangle, |01\rangle\}$, $\{|00\rangle, |10\rangle\}$, $\{|00\rangle, |01\rangle, |10\rangle\}$ and $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ are the reducible and indecomposable representations.

As $\lambda\lambda' \neq 1$, we obtain the second common eigenvector $|\psi\rangle$ of the generators $\hat{a}, \hat{d}$ which have the first common eigenvector $|00\rangle$,

$$|\psi\rangle = -|00\rangle + (1-\lambda\lambda')|11\rangle, \qquad \hat{a}|\psi\rangle = \hat{d}|\psi\rangle = \lambda\lambda'|\psi\rangle. \tag{37}$$

In the vector space spanned by $|01\rangle, |10\rangle$, a series of vectors $|\phi_n\rangle$ given by

$$|\phi_n\rangle = (\lambda')^n|01\rangle - \lambda^n|10\rangle, \qquad \hat{a}|\phi_n\rangle = |\phi_{n+1}\rangle, \qquad \hat{d}|\phi_n\rangle = -|\phi_{n+1}\rangle, \quad n \in \mathrm{N} \tag{38}$$

form a four-dimensional representation of the quantum algebra $\mathcal{A}_{-1}$ together with the vectors $|\psi\rangle$ and $|00\rangle$,

$$\hat{b}|\psi\rangle = -\hat{c}|\psi\rangle = -(1-\lambda\lambda')|\phi_1\rangle, \qquad \hat{b}|\phi_n\rangle = \hat{c}|\phi_n\rangle = ((\lambda')^n - \lambda^n)|00\rangle, \tag{39}$$

where $|00\rangle$ with any one of $|\phi_n\rangle$ forms a two-dimensional irreducible representation.

As $\lambda\lambda' = 1$, we define $|\psi_n\rangle = n|00\rangle + |11\rangle$, $n \in \mathrm{N}$, satisfying

$$\hat{a}|\psi_n\rangle = \hat{d}|\psi_n\rangle = |\psi_{n+1}\rangle, \qquad \hat{b}|\psi_n\rangle = -|\phi_1\rangle, \qquad \hat{c}|\psi_n\rangle = |\phi_1\rangle \tag{40}$$

which form a four-dimensional representation of $\mathcal{A}_{-1}$ with $|\phi_n\rangle$ and $|00\rangle$. See figure 1 where composition series can be easily recognized at the diagrammatical level.

4.3. *Four-dimensional representation:* $\underline{2}\otimes\underline{2}=\underline{2}\oplus\underline{2}$

In the second example, we obtain the four-dimensional representation $\underline{4}$ of $\mathcal{A}_{-1}$ via the coproduct construction in terms of the two-dimensional representation (22) of the generators $\hat{a}, \hat{b}, \hat{c}, \hat{d}$,

$$\begin{aligned} &\hat{a}|0\rangle = \lambda|0\rangle, \qquad \hat{a}|1\rangle = \lambda|1\rangle, \qquad \hat{d}|0\rangle = \lambda|1\rangle, \qquad \hat{d}|1\rangle = \lambda|0\rangle, \\ &\hat{b}|0\rangle = \mu|1\rangle, \qquad \hat{b}|1\rangle = -\mu|0\rangle, \qquad \hat{c}|0\rangle = \mu|0\rangle, \qquad \hat{c}|1\rangle = -\mu|1\rangle \end{aligned} \tag{41}$$

and the two-dimensional representation of the generators $\hat{a}', \hat{b}', \hat{c}', \hat{d}'$ which is the same as (22) except that λ, μ are replaced by λ', μ'. In addition, new symbols $z, \bar{z}, w, \bar{w}$ are introduced,

$$z = x - \mathrm{i}y, \qquad \bar{z} = x + \mathrm{i}y, \qquad w = v - \mathrm{i}u, \qquad \bar{w} = v + \mathrm{i}u, \tag{42}$$

where the symbols x, y, v, u denote the following products:

$$x = \lambda\lambda', \qquad y = \mu\mu', \qquad u = \lambda\mu', \qquad v = \mu\lambda'. \tag{43}$$

In this four-dimensional representation $\underline{4}$, the generator $\hat{d}$ has four eigenvectors denoted by four Dirac kets $|\chi_1\rangle, |\chi_2\rangle, |\tau_1\rangle, |\tau_2\rangle$ in terms of the Bell states $|\psi_\pm\rangle, |\phi_\pm\rangle$ (7), given by

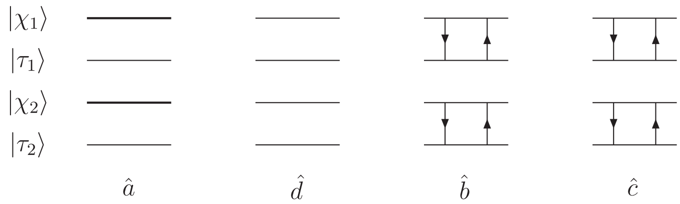

Figure 2. A four-dimensional representation of $\mathcal{A}_{-1}$: $\underline{2}\otimes\underline{2}=\underline{2}\oplus\underline{2}$.

$$
\begin{aligned}
&|\chi_1\rangle = \tfrac{1}{\sqrt{2}}(|\psi_+\rangle + \mathrm{i}|\phi_+\rangle), \qquad |\chi_2\rangle = \tfrac{1}{\sqrt{2}}(|\psi_-\rangle - \mathrm{i}|\phi_-\rangle),\\
&|\tau_1\rangle = \tfrac{1}{\sqrt{2}}(|\psi_-\rangle + \mathrm{i}|\phi_-\rangle), \qquad |\tau_2\rangle = \tfrac{1}{\sqrt{2}}(|\psi_+\rangle - \mathrm{i}|\phi_+\rangle),
\end{aligned} \tag{44}
$$

which are related to four distinct eigenvalues of the generator $\hat{d}$,

$$
\hat{d}|\chi_1\rangle = z|\chi_1\rangle, \qquad \hat{d}|\chi_2\rangle = -z|\chi_2\rangle, \qquad \hat{d}|\tau_1\rangle = -\bar{z}|\tau_1\rangle, \qquad \hat{d}|\tau_2\rangle = \bar{z}|\tau_2\rangle. \tag{45}
$$

The generator $\hat{a}$ has an eigenvalue z with two degenerate eigenvectors $|\chi_1\rangle, |\chi_2\rangle$ and another eigenvalue $\bar{z}$ with two degenerate eigenvectors $|\tau_1\rangle, |\tau_2\rangle$, i.e.,

$$
\hat{a}|\chi_1\rangle = z|\chi_1\rangle, \qquad \hat{a}|\chi_2\rangle = z|\chi_2\rangle, \qquad \hat{a}|\tau_1\rangle = \bar{z}|\tau_1\rangle, \qquad \hat{a}|\tau_2\rangle = \bar{z}|\tau_2\rangle. \tag{46}
$$

The Dirac kets $|\chi_1\rangle, |\tau_1\rangle$ form a two-dimensional *cyclic* representation $\underline{2}$ for the generators $\hat{b}, \hat{c}$:

$$
\hat{b}|\chi_1\rangle = -\bar{w}|\tau_1\rangle, \qquad \hat{b}|\tau_1\rangle = w|\chi_1\rangle, \qquad \hat{c}|\chi_1\rangle = \bar{w}|\tau_1\rangle, \qquad \hat{c}|\tau_1\rangle = w|\chi_1\rangle, \tag{47}
$$

and the Dirac kets $|\chi_2\rangle, |\tau_2\rangle$ form another two-dimensional *cyclic* representation $\underline{2}$ given by

$$
\hat{b}|\chi_2\rangle = \bar{w}|\tau_2\rangle, \qquad \hat{b}|\tau_2\rangle = -w|\chi_2\rangle, \qquad \hat{c}|\chi_2\rangle = \bar{w}|\tau_2\rangle, \qquad \hat{c}|\tau_2\rangle = w|\chi_2\rangle. \tag{48}
$$

Hence we reduce the four-dimensional representation $\underline{2}\otimes\underline{2}$ into the direct sum of two-dimensional representations: $\underline{2}\otimes\underline{2}=\underline{2}\oplus\underline{2}$. See figure 2 where states denoted by thick lines have the same eigenvalue and every irreducible representation $\underline{2}$ is *cyclic*.

With the bases of $|\chi_1\rangle, |\chi_2\rangle, |\tau_2\rangle, |\tau_1\rangle$, the four-dimensional representations of the generators $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ are given by the following matrices:

$$
\begin{aligned}
&\hat{a} = \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \otimes \mathbb{1}_2, \qquad \hat{b} = \begin{pmatrix} 0 & w \\ \bar{w} & 0 \end{pmatrix} \otimes \mathrm{i}\sigma_y,\\
&\hat{c} = \begin{pmatrix} 0 & w \\ \bar{w} & 0 \end{pmatrix} \otimes \sigma_x, \qquad \hat{d} = \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \otimes \sigma_z.
\end{aligned} \tag{49}
$$

These four Dirac kets $|\chi_1\rangle, |\chi_2\rangle, |\tau_1\rangle, |\tau_2\rangle$ are found to be local unitary transformations of the Bell states (7):

$$
\begin{aligned}
&|\chi_1\rangle = (\mathbb{1}_2 \otimes U_1)|\psi_+\rangle, \qquad |\chi_2\rangle = (\mathbb{1}_2 \otimes U_2)|\psi_+\rangle,\\
&|\tau_1\rangle = (\mathbb{1}_2 \otimes U_3)|\psi_+\rangle, \qquad |\tau_2\rangle = (\mathbb{1}_2 \otimes U_4)|\psi_+\rangle,
\end{aligned} \tag{50}
$$

where U_1, U_2, U_3, U_4 are unitary matrices given by

$$
\begin{aligned}
&U_1 = \tfrac{1}{\sqrt{2}}(\mathbb{1}_2 + \mathrm{i}\sigma_x), \qquad U_2 = \tfrac{1}{\sqrt{2}}(\sigma_z - \sigma_y),\\
&U_3 = \tfrac{1}{\sqrt{2}}(\sigma_z + \sigma_y), \qquad U_4 = \tfrac{1}{\sqrt{2}}(\mathbb{1}_2 - \mathrm{i}\sigma_x),
\end{aligned} \tag{51}
$$

and they form a set of complete orthonormal bases for 2×2 matrices,

$$
\operatorname{tr}\big(U_i^\dagger U_j\big) = 2\delta_{ij}, \qquad i, j = 1, 2, 3, 4. \tag{52}
$$

Hence, $|\chi_1\rangle, |\chi_2\rangle, |\tau_2\rangle, |\tau_1\rangle$ are maximally entangled states like the Bell states (7) because the entangled degree of a quantum state is invariant under local unitary transformations in quantum information [9].

5. Quantum algebra associated with the unitary evolution of Bell states

We study the quantum algebra associated with the unitary evolution of Bell states determined by Yang–Baxterization [27, 28].

5.1. Unitary evolution of Bell states via Yang–Baxterization

The YBE with the spectral parameter is called the quantum Yang–Baxter equation (QYBE). It has the form

$$\check{R}_i(x)\check{R}_{i+1}(xy)\check{R}_i(y) = \check{R}_{i+1}(y)\check{R}_i(xy)\check{R}_{i+1}(x) \tag{53}$$

with x or y the spectral parameter. It is well known that one can set up an integrable model by following a given recipe in terms of a solution of the QYBE, see [10, 11]. At $x = y = 0$, obviously, the QYBE reduces to $\check{R}_i\check{R}_{i+1}\check{R}_i = \check{R}_{i+1}\check{R}_i\check{R}_{i+1}$, the same as the braid group relation (2), $b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}$. Hence a solution $\check{R}(x)$ of the QYBE always reduces to a braid group representation $b = \check{R}(0)$. In other words, $\check{R}(0) = b$ can be regarded as the asymptotic condition of a solution $\check{R}(x)$ of the QYBE. Conversely, similar to a procedure of solving a differential equation with specified initial-boundary conditions, Baxterization [27] or Yang–Baxterization [28] represents a procedure of constructing a solution $\check{R}(x)$ of the QYBE (53) with the asymptotic condition, $\check{R}(0) = b$, where the braiding b-matrix has been specified. For example, for a b-matrix with two distinct eigenvalues λ_1 and λ_2, the corresponding $\check{R}(x)$-matrix obtained with Yang–Baxterization is found to be of the form

$$\check{R}(x) = b + x\lambda_1\lambda_2 b^{-1}. \tag{54}$$

Please refer to appendix A of [4] for more detail.

The Bell matrix $B_\pm$ forms a unitary braid representation, i.e., satisfying the braided YBE (3), see [6, 7] for the proof. Using Yang–Baxterization [3, 4], the corresponding $B_\pm(x)$ matrix as a solution of the QYBE (53) has the form

$$B_\pm(x) = \frac{1}{\sqrt{2(1+x^2)}}\begin{pmatrix} 1+x & 0 & 0 & q(1-x) \\ 0 & 1+x & \pm(1-x) & 0 \\ 0 & \mp(1-x) & 1+x & 0 \\ -q^{-1}(1-x) & 0 & 0 & 1+x \end{pmatrix}, \tag{55}$$

in which $B_\pm = B_\pm(0)|_{q=1}$, q called the deformation parameter[5].

In terms of new variables of angles θ and φ,

$$\cos\theta = \frac{1}{\sqrt{1+x^2}}, \qquad \sin\theta = \frac{x}{\sqrt{1+x^2}}, \qquad q = \mathrm{e}^{-\mathrm{i}\varphi}, \tag{56}$$

we rewrite the $B_\pm(x)$ matrix into an exponential function,

$$B_\pm(\theta) = \cos\left(\frac{\pi}{4} - \theta\right) + 2\mathrm{i}\sin\left(\frac{\pi}{4} - \theta\right)H_\pm = \mathrm{e}^{\mathrm{i}(\frac{\pi}{2} - 2\theta)H_\pm}, \tag{57}$$

where the symbol $H_\pm$ called the Hamiltonian [3, 4] has the form

$$\begin{aligned} &H_+ = \tfrac{1}{2}\sigma_{n_1} \otimes \sigma_{n_2}, \qquad H_- = \tfrac{1}{2}\sigma_{n_2} \otimes \sigma_{n_1}, \\ &\sigma_{n_1} = \sigma_+\,\mathrm{e}^{-\frac{\mathrm{i}}{2}(\varphi+\pi)} + \sigma_-\mathrm{e}^{\frac{\mathrm{i}}{2}(\varphi+\pi)}, \qquad \sigma_{n_2} = \sigma_+\,\mathrm{e}^{-\frac{\mathrm{i}}{2}\varphi} + \sigma_-\mathrm{e}^{\frac{\mathrm{i}}{2}\varphi}. \end{aligned} \tag{58}$$

[5] Note that the $B_\pm(x)$ matrix satisfies the free fermion condition [23] and therefore is a special case of the free fermion $\check{R}$-matrix known from [24].

The unitary evolution of the Bell states under the Hamiltonian $H_\pm$ with the time variable θ is determined by the unitary $B_\pm(\theta)$-matrix, namely,

$$\begin{aligned}
B_\pm(\theta)|00\rangle &= \cos\left(\frac{\pi}{4}-\theta\right)|00\rangle - \mathrm{e}^{\mathrm{i}\varphi}\sin\left(\frac{\pi}{4}-\theta\right)|11\rangle,\\
B_\pm(\theta)|11\rangle &= \mathrm{e}^{-\mathrm{i}\varphi}\sin\left(\frac{\pi}{4}-\theta\right)|00\rangle + \cos\left(\frac{\pi}{4}-\theta\right)|11\rangle,\\
B_\pm(\theta)|01\rangle &= \cos\left(\frac{\pi}{4}-\theta\right)|01\rangle \mp \sin\left(\frac{\pi}{4}-\theta\right)|10\rangle,\\
B_\pm(\theta)|10\rangle &= \pm\sin\left(\frac{\pi}{4}-\theta\right)|01\rangle) + \cos\left(\frac{\pi}{4}-\theta\right)|10\rangle.
\end{aligned} \tag{59}$$

In the literature, there is a long history of dealing with the eight-vertex models [11, 12], the Sklyanin algebra [34, 35] and some relevant extensions including Potts models [12, 36]. Solutions of the YBE with the spectral parameter, i.e., symmetric solutions of eight-vertex models, have double-period elliptic functions [11, 12] as matrix entries. However, they are nothing with the recent study in quantum information. As a matter of fact, we find standard methodologies for six-vertex models [12, 13] helpful for the purpose of exploring applications of the Bell matrix to quantum information. For example, Yang–Baxterization of the Bell matrix [3, 4] has matrix entries in terms of single-period trigonometric functions which characterize six-vertex models.

5.2. Quantum algebra associated with the $B_+(x)$-matrix

The $B_\pm(x)$-matrix (55) is obtained via Yang–Baxterization of the Bell matrix $B_\pm$, see [3, 4] for the detail, and is a solution of the YBE with the spectral parameter x. With the help of the FRT construction [26], there also exists a quantum algebra determined by the following $\check{R}TT$ relation:

$$\check{R}(xy^{-1})(T(x)\otimes T(y)) = (T(y)\otimes T(x))\check{R}(xy^{-1}), \tag{60}$$

which is invariant under the rescaling transformation of the $\check{R}(x)$-matrix and $T(x)$-matrix by global scalar factors. Here the $\check{R}(x)$-matrix has the form of the $B_\pm(x)$-matrix (55) without the normalization factor. Assume the $T_\pm(x)$-matrix to have a formalism similar to $B_\pm(x)$ which is a linear combination between the Bell matrix $B_\pm$ (9) and its inverse $B_\pm^{-1}$,

$$B_\pm(x) = B_\pm + 2xB_\pm^{-1}, \qquad T_\pm(x) = T_\pm + 2xT'_\pm, \tag{61}$$

where the $B_\pm$ matrix does not have the normalization factor $1/\sqrt{2}$ in (9), the $T_\pm$-matrix has four noncommutative operators $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ as its matrix entries and the $T'_\pm$-matrix has four noncommutative operators $\hat{a}', \hat{b}', \hat{c}', \hat{d}'$ as its matrix entries. The original $\check{R}TT$ relation (60) with the spectral parameters x, y is simplified into the four $\check{R}TT$ relations independent of the spectral parameter,

$$\begin{aligned}
B_\pm(T_\pm\otimes T_\pm) &= (T_\pm\otimes T_\pm)B_\pm, & B_\pm(T'_\pm\otimes T'_\pm) &= (T'_\pm\otimes T'_\pm)B_\pm,\\
B_\pm(T_\pm\otimes T'_\pm) &= (T'_\pm\otimes T_\pm)B_\pm, & B_\pm(T'_\pm\otimes T_\pm) &= (T_\pm\otimes T'_\pm)B_\pm.
\end{aligned} \tag{62}$$

In the following, for simplicity, we only consider the quantum algebra determined by the $\check{R}TT$ relations in terms of the B_+, T_+ and T'_+ matrices.

Obviously, the generators $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ and $\hat{a}', \hat{b}', \hat{c}', \hat{d}'$ satisfy the same quantum algebra $\mathcal{A}_{-1}$. The algebraic equation $B_+(T_+\otimes T'_+) = (T'_+\otimes T_+)B_+$ leads to the following algebraic relations:

$$\begin{aligned}
&[a, a'] = -qcc' - q^{-1}b'b, && \{a, b'\} = a'b - qcd',\\
&\{a, c'\} = ca' + q^{-1}d'b, && [a, d'] = cb' - c'b,\\
&[b, a'] = b'a - qdc', && [b, b'] = -qdd' + qa'a,\\
&\{b, c'\} = da' - d'a, && \{b, d'\} = db' - qc'a,\\
&[c, a'] = -ac' - q^{-1}b'd, && \{c, b'\} = a'd - ad',\\
&[c, c'] = q^{-1}aa' - q^{-1}d'd, && \{c, d'\} = c'd + q^{-1}ab',\\
&[d, a'] = b'c - bc', && [d, b'] = qa'c - bd',\\
&[d, c'] = d'c + q^{-1}ba', && [d, d'] = q^{-1}bb' + qc'c,
\end{aligned} \tag{63}$$

while the algebraic equation $B_+(T'_+ \otimes T_+) = (T_+ \otimes T'_+)B_+$ leads to more constraint algebraic relations:

$$\begin{aligned}
&[a, a'] = q^{-1}bb' + qc'c, && [a, b'] = -ba' + qd'c,\\
&[a, c'] = q^{-1}bd' + a'c, && [a, d'] = -bc' + b'c,\\
&\{b, a'\} = ab' - qc'd, && [b, b'] = -qaa' + qd'd,\\
&\{b, c'\} = ad' - a'd, && [b, d'] = -qac' + b'd,\\
&\{c, a'\} = q^{-1}db' + c'a, && \{c, b'\} = -da' + d'a,\\
&[c, c'] = q^{-1}dd' - q^{-1}a'a, && [c, d'] = -dc' - q^{-1}b'a,\\
&[d, a'] = cb' - c'b, && \{d, b'\} = -qca' + d'b,\\
&\{d, c'\} = cd' + q^{-1}a'b, && [d, d'] = -qcc' - q^{-1}b'b.
\end{aligned} \tag{64}$$

The algebraic relations (63) determined by $B_+(T_+ \otimes T'_+) = (T'_+ \otimes T_+)B_+$ have the simplified forms:

$$\begin{aligned}
&[\hat{a}, \hat{a}'] = -q\hat{c}\hat{c}' - q^{-1}\hat{b}'\hat{b}, && [\hat{b}, \hat{b}'] = -q\hat{d}\hat{d}' + q\hat{a}'\hat{a},\\
&[\hat{a}, \hat{a}'] = [\hat{d}', \hat{d}], && [\hat{b}, \hat{b}'] = q^2[\hat{c}, \hat{c}'],\\
&\{a, b'\} = a'b - qcd', && [b, a'] = b'a - qdc',\\
&\{a, b'\} = q[d, c'], && [b, a'] = -q\{d', c\}\\
&\{\hat{a}, \hat{c}'\} = \hat{c}\hat{a}' + q^{-1}\hat{d}'\hat{b}, && [\hat{c}, \hat{a}'] = -\hat{a}\hat{c}' - q^{-1}\hat{b}'\hat{d},\\
&\{\hat{a}, \hat{c}'\} = q^{-1}[\hat{d}, \hat{b}'], && [\hat{c}, \hat{a}'] = -q^{-1}\{\hat{b}, \hat{d}'\}\\
&[\hat{a}, \hat{d}'] = \hat{c}\hat{b}' - \hat{c}'\hat{b}, && \{\hat{b}, \hat{c}'\} = \hat{d}\hat{a}' - \hat{d}'\hat{a},\\
&[\hat{a}, \hat{d}'] = [\hat{a}', \hat{d}], && \{\hat{b}, \hat{c}'\} = \{\hat{c}, \hat{b}'\}
\end{aligned} \tag{65}$$

and those algebraic relations (64) from $\check{R}(T'_+ \otimes T_+) = (T_+ \otimes T'_+)\check{R}$ can be also simplified,

$$\begin{aligned}
&[\hat{a}, \hat{a}'] = q^{-1}\hat{b}\hat{b}' + q\hat{c}'\hat{c}, && [\hat{b}, \hat{b}'] = -q\hat{a}\hat{a}' + q\hat{d}'\hat{d},\\
&[\hat{a}, \hat{a}'] = [\hat{d}', \hat{d}], && [\hat{b}, \hat{b}'] = q^2[\hat{c}, \hat{c}'],\\
&[\hat{a}, \hat{b}'] = -\hat{b}\hat{a}' + q\hat{d}'\hat{c}, && \{\hat{a}', \hat{b}\} = \hat{a}\hat{b}' - q\hat{c}'\hat{d},\\
&[\hat{a}, \hat{b}'] = q\{\hat{d}, \hat{c}'\}, && \{\hat{a}', \hat{b}\} = q[\hat{d}', \hat{c}],\\
&[\hat{a}, \hat{c}'] = q^{-1}\hat{b}\hat{d}' + \hat{a}'\hat{c}, && \{\hat{c}, \hat{a}'\} = q^{-1}\hat{d}\hat{b}' + \hat{c}'\hat{a},
\end{aligned}$$

$$
\begin{aligned}
&[\hat{a}, \hat{c}'] = q^{-1}\{\hat{d}, \hat{b}'\}, && \{\hat{c}, \hat{a}'\} = -q^{-1}[\hat{b}, \hat{d}'],\\
&[\hat{a}, \hat{d}'] = -\hat{b}\hat{c}' + \hat{b}'\hat{c}, && \{\hat{b}, \hat{c}'\} = \hat{a}\hat{d}' - \hat{a}'\hat{d},\\
&[\hat{a}, \hat{d}'] = [\hat{a}', \hat{d}], && \{\hat{b}, \hat{c}'\} = \{\hat{c}, \hat{b}'\}.
\end{aligned} \tag{66}
$$

After some further algebraic reductions, two types of generators $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ and $\hat{a}', \hat{b}', \hat{c}', \hat{d}'$ of the quantum algebra $\mathcal{B}_{-1}$ are found to satisfy commutative relations,

$$
\begin{aligned}
&\hat{a}\hat{a}' = \hat{a}'\hat{a}, && \hat{d}\hat{d}' = \hat{d}'\hat{d}, && \hat{b}\hat{b}' = \hat{b}'\hat{b}, && \hat{c}\hat{c}' = \hat{c}\hat{c}',\\
&\hat{a}\hat{b}' = \hat{a}'\hat{b}, && \hat{b}'\hat{a} = \hat{b}\hat{a}', && \hat{d}'\hat{c} = \hat{d}\hat{c}', && \hat{c}\hat{d}' = \hat{c}'\hat{d},\\
&\hat{a}\hat{c}' = \hat{a}'\hat{c}, && \hat{c}'\hat{a} = \hat{c}\hat{a}', && \hat{b}\hat{d}' = \hat{b}'\hat{d}, && \hat{d}\hat{b}' = \hat{d}'\hat{b},\\
&\hat{a}'\hat{d} = \hat{a}\hat{d}', && \hat{d}\hat{a}' = \hat{d}'\hat{a}, && \hat{b}\hat{c}' = \hat{b}'\hat{c}, && \hat{c}\hat{b}' = \hat{c}'\hat{b},
\end{aligned} \tag{67}
$$

and additional algebraic relations similar to those of (19) determining $\mathcal{A}_{-1}$,

$$
\begin{aligned}
&\hat{a}'\hat{a} = \hat{d}\hat{d}', && \hat{a}\hat{d}' = \hat{d}'\hat{a}, && \hat{b}\hat{b}' = -q^2\hat{c}'\hat{c},\\
&\hat{b}\hat{c}' = -\hat{c}'\hat{b}, && \hat{a}'\hat{b} = q\hat{d}'\hat{c}, && \hat{b}\hat{a}' = -q\hat{c}\hat{d}',
\end{aligned} \tag{68}
$$

where the quantum algebra determined by these relations is called the algebra $\mathcal{B}_{-1}$ over the complex field $\mathbb{C}$. The deformation parameter q is irrelevant due to the rescaling transformation: $q\hat{c} \to \hat{c}$ and $q\hat{c}' \to \hat{c}'$.

Note that further efforts will be needed to obtain interesting algebraic structures in the representation of the quantum algebra $\mathcal{B}_{-1}$ similar to what we have found in the quantum algebra $\mathcal{A}_{-1}$. In addition, the quantum algebra $\mathcal{B}_{-1}$ is a quotient algebra of the FRT dual algebra that Arnaudon *et al* [30–32] have discussed.

6. Concluding remarks and outlooks

The Bell matrix is an interdisciplinary subject to explore the connections between the YBE and quantum information theory as well as compare topological entanglements and quantum entanglements. In this paper, we study the quantum algebra $\mathcal{A}_{-1}$ associated with the Bell matrix. We find that in its four-dimensional representation there are a composition series over the complex field $\mathbb{C}$ and the representation $\underline{2} \otimes \underline{2} = \underline{2} \oplus \underline{2}$. The latter one is not true in the representation of the Lie algebra [13]. Besides, we show the quantum algebra $\mathcal{B}_{-1}$ associated with the unitary evolutions of the Bell states.

There still remain interesting unsolved problems in the project of exploring algebraic structures related to the Bell matrix. For example, (i) does $\mathcal{A}_{-1}$ have a four-dimensional representation: $\underline{2} \otimes \underline{2} = \underline{1} \oplus \underline{3}$? (ii) the construction of universal $\check{R}$-matrix [13] in terms of generators of $\mathcal{A}_{-1}$; (iii) interesting algebraic structures underlying $\mathcal{B}_{-1}$ such as the quantum double; (iv) new quantum algebras obtained by applying methodologies for deriving the Sklyanin algebra [34, 35] to the Bell matrix.

About physical realizations of the Bell matrix and quantum algebras $\mathcal{A}_{-1}$ and $\mathcal{B}_{-1}$, they cannot give rise to an ordinary integrable spin–chain model. Because the Bell matrix and its Yang–Baxterization have negative entries which cannot be explained as positive (zero) Boltzmann weights in statistical physics. We choose to study their physical applications in the quantum information theory. In quantum computation [2–4, 6, 7], the Bell matrix B_+ (B_-) and the permutation matrix P can be respectively recognized as a universal quantum gate and the swap gate. Moreover, P, B_+ (B_-) yield a representation of the virtual braid group which is a possible language for universal quantum computing [8, 37].

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Note added. (on our further research based on the present paper). In [37], the high-dimensional Bell matrices associated with the GHZ states (a generalization of the Bell states) have been constructed in terms of the almost-complex structure. In [38], the Bell matrix as a unitary braid representation in terms of the representation of the extraspecial two-groups provides a possible link between the quantum error correction and topological quantum computing.

References

[1] Dye H A 2003 Unitary solutions to the Yang–Baxter equation in dimension four *Quant. Inf. Proc.* **2** 117–50
[2] Kauffman L H and Lomonaco S J (Jr) 2004 Braiding operators are universal quantum gates *New J. Phys.* **6** 134
[3] Zhang Y, Kauffman L H and Ge M L 2005 Universal quantum gate, Yang–Baxterization and Hamiltonian *Int. J. Quant. Inform.* **3** (4)
[4] Zhang Y, Kauffman L H and Ge M L 2005 Yang–Baxterizations, universal quantum gates and Hamiltonians *Quant. Inf. Proc.* **4** 159–97
[5] Franko J, Rowell E C and Wang Z 2006 Extraspecial 2-groups and images of Braid group representations *J. Knot Theory Ramifications* **15** 413–28
[6] Zhang Y 2006 Teleportation, Braid group and Temperley–Lieb algebra *J. Phys. A: Math. Gen.* **39** 11599–622
[7] Zhang Y 2006 Algebraic structures underlying quantum information protocols *Preprint* quant-ph/0601050
[8] Zhang Y, Kauffman L H and Werner R F 2006 Permutation and its partial transpose *Int. J. Quant. Inform.* **5** 469–507 (*Preprint* quant-ph/0606005)
[9] Nielsen M and Chuang I 1999 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)
[10] Yang C N 1967 Some exact results for the many body problems in one dimension with repulsive delta function interaction *Phys. Rev. Lett.* **19** 1312–4
[11] Baxter R J 1972 Partition function of the eight-vertex lattice model *Ann. Phys.* **70** 193–228
[12] Faddeev L D 1987 Lectures on quantum inverse scattering method *Integrable Systems (Nankai Lectures on Mathematical Physics)* (Singapore: World Scientific) pp 23–70
[13] Kassel C 1995 *Quantum Groups* (New York: Springer)
[14] Werner R F 1989 Quantum states with Einstein–Podolsky–Rosen correlations admitting a hidden-variable model *Phys. Rev.* A **40** 4277
[15] Kauffman L H 2002 *Knots and Physics* (Singapore: World Scientific)
[16] Aravind P K 1997 Borromean entanglement of the GHZ state *Potentiality, Entanglement and Passion-at-a-Distance* ed R S Cohen, S Robert, H Michael and S John (Boston: Kluwer Academic)
[17] Kitaev A Yu 2003 Fault-tolerant quantum computation by anyons *Ann. Phys.* **303** 2 (*Preprint* quant-ph/9712048)
[18] Freedman M, Kitaev A, Larsen M and Wang Z 2003 Topological quantum computation *Bull. Am. Math. Soc., N. S.* **40** 31–8
[19] Sarma S Das, Freedman M, Nayak C, Simon S and Stern A 2007 Non-Abelian anyons and topological quantum computation *Preprint* 0707.1889
[20] Ekert A K 1991 Quantum cryptography based on Bell's theorem *Phys. Rev. Lett.* **67** 661–3
[21] Bennett C H, Brassard G, Crepeau C, Jozsa R, Peres A and Wootters W K 1993 Teleporting an unknown quantum state via dual classical and Einstein–Podolsky–Rosen channels *Phys. Rev. Lett.* **70** 1895–9
[22] Bell J S 1964 On the Einstein–Podolsky–Rosen paradox *Physics* **1** 195–200
[23] Fan C and Wu F Y 1970 General lattice model of phase transitions *Phys. Rev.* B **2** 723-733
[24] Felderhof B U 1973 Direct diagonalization of the transfer matrix of the zero field free-Fermion model *Physica* **65** 421–51
[25] Jing N, Ge M L and Wu Y-S 1991 New quantum group associated with a 'nonstandard' Braid group representation *Lett. Math. Phys.* **21** 183

[26] Reshetikhin N Yu, Takhtadjian L A and Faddeev L D 1989 Quantization of Lie groups and Lie algebras *Algebra Analiz* **1** 178–206
Reshetikhin N Yu, Takhtadjian L A and Faddeev L D 1990 Quantization of Lie groups and Lie algebras *Leningrad Math. J.* **1** 193–225 (Engl. Transl.)
[27] Jones V F R 1991 Baxterization *Int. J. Mod. Phys.* A **6** 2035–43
[28] Ge M L, Xue K and Wu Y-S 1991 Explicit trigonometric Yang–Baxterization *Int. J. Mod. Phys.* A **6** 3735
[29] Kauffman L H and Radford D E 2003 Bi-oriented quantum algebras, and a generalized Alexander polynomial for virtual links *Contemp. Math.* **318** 113–40
[30] Arnaudon D, Chakrabarti A, Dobrev V K and Mihov S G 2002 Duality and representations for new exotic bialgebras *J. Math. Phys.* **43** 6238–64
[31] Arnaudon D, Chakrabarti A, Dobrev V K and Mihov S G 2003 Spectral decomposition and Baxterisation of exotic bialgebras and associated noncommutative geometries *Int. J. Mod. Phys.* A **18** 4201–13
[32] Arnaudon D, Chakrabarti A, Dobrev V K and Mihov S G 2006 Exotic bialgebra S03: representations, Baxterisation and applications *Ann. Henri Poincare* (to appear) (*Preprint* math.QA/0601708)
[33] Hietarinta J 1993 Solving the two-dimensional constant quantum Yang–Baxter equation *J. Math. Phys.* **34** 1725
[34] Sklyanin E K 1982 Some algebraic structures connected with the Yang–Baxter equation *Funct. Anal. Appl.* **16** 27–34
[35] Sklyanin E K 1983 Some algebraic structures connected with the Yang–Baxter equation: representations of quantum algebras *Funct. Anal. Appl.* **17** 34–48
[36] Bajhanov V V and Stroganov Yu G 1989 Chiral Potts model as a descendant of the six-vertex model *YBE in Integral Systems* ed M Jimbo (Singapore: World Scientific) pp 673–98
[37] Zhang Y and Ge M L 2007 GHZ states, almost-complex structure and Yang–Baxter equation *Quant. Inf. Proc.* **6** 363–79
[38] Zhang Y, Rowell E C, Wu Y-S, Wang Z and Ge M L 2007 From extraspecial two-groups to GHZ states *Preprint* 0706.1761